AXISYMMETRIC THERMOELASTICITY PROBLEM FOR A CYLINDER WITH A SLOT

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UDC 539.377

The axisymmetric thermoelastic state of an isotropic,circular, infinite cylinder with an external annular groove is considered. It is assumed that uniformly distributed heat sources act on part of the slot surface, while the side surface of the cylinder is heat insulated. A formula is obtained to determine the normal stresses in the plane of the slot.

1. Let us consider a cylindrical r, φ, z coordinate system whose z axis coincides with the longitudinal axis of an infinite solid cylinder with an external annular slot ($b \le r \le R$) in the $z=0$ plane (Fig. 1). Let uniformly distributed heat sources with intensity W_0 act on a part of the slot surface ($a \le r \le R$, $a > b$). It is assumed that the side surface of the cylinder is heat insulated, free of tangential stresses, and fixed so that its points have no radial displacements.

Taking account of the symmetry condition relative to the $z=0$ plane, let us examine the action of the heat sources distributed uniformly over the annular domain $a \le r \le R$, $z=0$. In this case the temperature function satisfying the condition

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} = 0 \tag{1.1}$$

is a solution of the equation

$$\Delta T = -\frac{W_0}{\lambda}\delta(z)\eta(r-a), \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{1.2}$$

where λ is the heat conduction coefficient, $\delta(z)$ is a δ-function,

$$\eta(r-\alpha) = 1 \quad (r > a), \qquad \eta(r-a) = 0 \quad (r < a)$$

Using the integral representation

$$\delta(z) = \frac{1}{\pi}\int_0^\infty \cos\lambda z\, d\lambda$$

and the Bessel function expansion

$$\eta(r-a) = -2\frac{a}{R}\sum_{n=1}^\infty \frac{J_1(\lambda_n a/R)\, J_0(\lambda_n r/R)}{\lambda_n J_0^2(\lambda_n)} \tag{1.3}$$

we find a solution of (1.2) in the form

$$T(r,z) = -\frac{W}{\lambda}\alpha R\sum_{n=1}^\infty \frac{J_1(\lambda_n\alpha)\, J_0(\lambda_n\rho)}{\lambda_n^2 J_0^2(\lambda_n)} e^{-\lambda_n\xi} \tag{1.4}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 154-158, November-December, 1971. Original article submitted October 12, 1970.

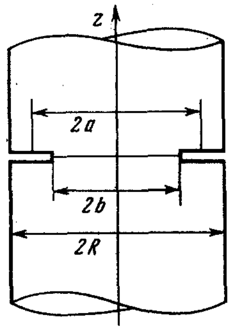

Fig. 1

Here $\alpha = a/R$, $\rho = r/R$, $\xi = z/R$, $\lambda_n (n=1, 2, 3, \ldots)$ are positive roots of the equation $J_0'(\lambda_n) = 0$ in order of increasing magnitude.

The stresses $\sigma_{ij}^{(1)}$ and displacements $u_i^{(1)}$ in the semiinfinite cylinder $0 \le z < \infty$, $0 \le r \le R$ due to this temperature field (1.4) can be found by using the thermoelastic potential Φ satisfying the equation [1]

$$\Delta\Phi = mT, \qquad m = \frac{1+\nu}{1-\nu}\alpha_T \tag{1.5}$$

where α_T is the coefficient of linear expansion of the material. Taking account of (1.4), let us write the solution of (1.5) as

$$\Phi(r, z) = \frac{mW_0R^3}{2\lambda}\alpha \sum_{n=1}^{\infty} \frac{J_1(\lambda_n\alpha) J_0(\lambda_n\rho)}{\lambda_n^4 J_0^2(\lambda_n)} l^{-\lambda_n\xi}(1+\lambda_n\xi) \tag{1.6}$$

The stress and displacement components corresponding to the thermoelastic potential (1.6) will satisfy the following conditions on the side surface $r = R$ and the endface $z = 0$ of the cylinder:

$$\sigma_{rz}^{(1)}(R, z) = 0, \quad u_r^{(1)}(R, z) = 0 \quad \text{for} \quad 0 \leqslant z < \infty \tag{1.7}$$

$$\sigma_{rz}^{(1)}(r, 0) = 0, \quad \sigma_{zz}^{(1)}(r, 0) = -GmT(r, 0) \quad \text{for} \quad 0 \leqslant r \leqslant R \tag{1.8}$$

In order for the edges of the annular slot $b \le r \le R$, $z = 0$ to be free of loads, it is necessary to impose a state with the components $\sigma_{ij}^{(2)}$, $u_i^{(2)}$ which satisfy the conditions

$$\sigma_{zz}^{(2)}(r, 0) = -\sigma_{zz}^{(1)}(r, 0), \quad b \leqslant r < R, \quad u_z^{(2)}(r, 0) = 0, \quad 0 \leqslant r < b \tag{1.9}$$

$$\sigma_{rz}^{(2)}(r, 0) = 0, \qquad 0 \leqslant r \leqslant R \tag{1.10}$$

$$\sigma_{rz}^{(2)}(R, z) = 0, \qquad u_r^{(2)}(R, z) = 0, \qquad 0 \leqslant z < \infty \tag{1.11}$$

on the stress-strain state $\sigma_{ij}^{(1)}$, $u_i^{(1)}$.

Since the tangential stresses $\sigma_{rz}^{(2)}$ equal zero in the $z = 0$ plane, the displacements and stresses may then be expressed in terms of just one harmonic axisymmetric function $\varphi(r, z)$ [2]

$$u_r^{(2)} = z\frac{\partial^2\varphi}{\partial r\partial z} + (1-2\nu)\frac{\partial\varphi}{\partial r}, \qquad \sigma_{rz}^{(2)} = 2Gz\frac{\partial^3\varphi}{\partial r\partial z^2} \tag{1.12}$$

Here G is the shear modulus of the cylinder material.

Selecting the harmonic function $\varphi(r, z)$ in the form

$$\varphi(r, z) = \frac{R^2}{2(1-\nu)} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} B_n J_0(\lambda_0\rho) e^{-\lambda_n\xi} \tag{1.13}$$

we find that the conditions (1.10) and (1.11) are satisfied identically. Satisfying the conditions (1.9), we obtain dual series equations

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} B_n J_n(\lambda_n\rho) = 0, \qquad 0 \leqslant \rho < \beta \tag{1.14}$$

$$\sum_{n=1}^{\infty} B_n J_0(\lambda_n\rho) = (1-\nu)m\frac{W_0a}{\lambda}\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\alpha)J_0(\lambda_n\rho)}{\lambda_n^2 J_0^2(\lambda_n)}, \qquad \beta < \rho < 1 \tag{1.15}$$

where

$$\beta = b/R$$

2. The solution of the dual equations of the form (1.14), (1.15) has been obtained in [4, 5]. The problem of the impression of a stamp in the endfaces of a semiinfinite cylinder [3] also reduced to analogous equations.

Using the relationship

$$B_n = (1-\nu)\frac{mW_0a}{\lambda}\frac{J_1(\lambda_n\alpha)}{\lambda_n^2 J_0^2(\lambda_n)} + B_n^* \tag{2.1}$$

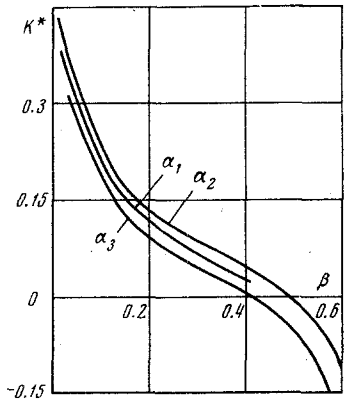

Fig. 2

let us represent (1.14) and (1.15) as

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} B_n^* J_n(\lambda_n\rho) = -(1-\nu)\frac{mW_0 a}{\lambda} \sum_{n=1}^{\infty} \frac{J_1(\lambda_n\alpha)}{\lambda_n^3 J_0^2(\lambda_n)} J_0(\lambda_n\rho) \tag{2.2}$$

$$\sum_{n=1}^{\infty} B_n^* J_0(\lambda_n\rho) = 0, \qquad \beta < \rho \leqslant 1 \tag{2.3}$$

Following [5], we obtain

$$\sum_{n=1}^{\infty} B_n^* J_0(\lambda_n\rho) = -\frac{1}{\rho}\frac{d}{d\rho}\int_{\rho}^{\beta} \frac{tg(t)\,dt}{\sqrt{t^2-\rho^2}}, \qquad 0 \leqslant \rho < \beta \tag{2.4}$$

By means of the formula defining the coefficients of the Dini expansion we find

$$B_n^* = \frac{2}{J_0^2(\lambda_n)} \int_0^{\beta} g(t)\cos\lambda_n t\,dt \tag{2.5}$$

Substituting the expression for B_n^* into (2.2) and taking into account [5]

$$2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n\rho)\cos\lambda_n t}{\lambda_n J_0^2(\lambda_n)} = -L(t,\rho) + \begin{cases} 0, & \rho < t \\ (\rho^2-t^2)^{-1/2}, & \rho > t \end{cases}$$

$$L(t,\rho) = 2\sqrt{1-t^2} + \frac{2}{\pi}\int_0^{\infty} \frac{K_1(\xi)}{\xi I_1(\xi)} \operatorname{ch} t\xi\,[2I_1(\xi) - \xi I_0(\rho\xi)]\,d\xi \tag{2.6}$$

we obtain a Fredholm integral equation of the second kind to determine the function g(t) after appropriate manipulation [5]:

$$g(t) = \int_0^{\beta} g(u)K(u,t)\,du - (1-\nu)\,m\frac{W_0 a}{\lambda}\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\alpha)\cos\lambda_n t}{\lambda_n^3 J_0^2(\lambda_n)}$$

$$K(u,t) = \frac{2}{\pi}\frac{d}{dt}\int_0^{t}\frac{\rho L(u,\rho)}{\sqrt{t^2-\rho^2}}\,d\rho = \frac{4}{\pi} + \frac{4}{\pi^2}\int_0^{\infty}\frac{K_1(\xi)}{\xi I_1(\xi)}\,[2I_1(\xi) - \xi\operatorname{ch}(t\xi)\operatorname{ch}(u\xi)]\,d\xi \tag{2.7}$$

Here $I_1(\xi)$, $K_1(\xi)$ are the modified Bessel functions of first and second order, respectively.

To evaluate the sum in the right side of (2.7), let us multiply (1.3) by $\rho(t^2-\rho^2)^{-1/2}$ and let us integrate with respect to ρ between 0 and t.

We obtain

$$2\alpha\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\alpha)\sin\lambda_n t}{\lambda_n^2 J_0^2(\lambda_n)} = \begin{cases}\sqrt{t^2-\alpha^2} & (t>\alpha)\\ t & (t<\alpha)\end{cases}$$

Integrating the last expression with respect to t, we find

$$2\alpha\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\alpha)\cos\lambda_n t}{\lambda_n^3 J_0^2(\lambda_n)} = A(\alpha) - \frac{t^2}{2}$$

$$A(\alpha) = 2\alpha\sum_{n=1}^{\infty}\frac{J_1(\lambda_n\alpha)}{\lambda_n^3 J_0^2(\lambda_n)} \tag{2.8}$$

Therefore, (2.7) becomes

$$g(t) = \int_0^{\beta} g(u)K(u,t)\,du - (1-\nu)\frac{W_0 R}{\pi\lambda}\left[A(\alpha) - \frac{t^2}{2}\right] \tag{2.9}$$

The kernel K(u, t) of (2.9) can be represented as

$$K(u,t) = \sum_{m=0}^{\infty} b_{2m}(u)\,t^{2m}$$

$$b_0(u) = \frac{4}{\pi} + \frac{4}{\pi^2}\left[T^* - \sum_{s=1}^{\infty}\frac{T_{2s}u^{2s}}{(2s)!}\right]$$

$$b_{2m}(u) = -\frac{4}{\pi^2}\sum_{s=1}^{\infty}\frac{T_{2m+2s-2}u^{2s-2}}{(2m)!\,(2s-2)!} \qquad (m = 1, 2, \ldots)$$

$$T^* = \sum_{s=1}^{\infty}\frac{T_{2s}}{2^{2s}s!\,(s+1)!}, \qquad T_n = \int_0^{\infty}\frac{K_1(\xi)}{I_1(\xi)}\xi^n d\xi \tag{2.10}$$

Numerical values of the coefficients T_n are presented in [4].

Taking account of the expansion (2.10), the solution of the integral equation (2.9) can be sought as

$$g(t) = \sum_{m=0}^{\infty} Q_{2m}t^{2m} \tag{2.11}$$

where the coefficients Q_{2m} are determined from the infinite system

$$Q_{2m} = \sum_{k=0}^{\infty} Q_{2k}c_{2k,\,2m} - (1-\nu)\,m\frac{W_0 R}{\pi\lambda}\left[A(\alpha)\,\delta_m{}^{0} - \frac{1}{2}\delta_m{}^{1}\right] \qquad (m = 0, 1, 2, \ldots)$$

$$c_{2k,\,0} = \frac{4}{\pi}\left(1+\frac{T^*}{\pi}\right)\frac{\beta^{2k+1}}{2k+1} - \frac{4}{\pi^2}\sum_{s=1}^{\infty}\frac{T_{2s}\beta^{2s+2k+1}}{(2s)!\,(2s+2k+1)} \qquad (k = 0, 1, 2, \ldots)$$

$$c_{2k,\,2m} = -\frac{4}{\pi^2}\sum_{s=1}^{\infty}\frac{T_{2m+2s-2}\beta^{2k+2s-1}}{(2m)!\,(2s-2)\,(2k+2s-1)} \qquad (k = 0, 1, 2, \ldots;\ m = 1, 2, \ldots)$$

$$\delta_m{}^{j} = \begin{cases} 1, & m = j \\ 0, & m \neq j \end{cases} \tag{2.12}$$

Using (1.12), (2.1), and (2.4), we find the normal stresses in the z = 0 plane

$$\sigma_{zz}^{(2)}(r, 0) = GmT(\rho, 0) + \frac{G}{1-\nu}\frac{1}{\rho}\frac{d}{d\rho}\int_{\rho}^{\beta}\frac{tg(t)\,dt}{\sqrt{t^2-\rho^2}}, \qquad 0 \leqslant \rho < \beta \tag{2.13}$$

By using (1.8), (2.11), (2.13), we determine the coefficient of stress intensity for the annular slot

$$K = \sqrt{2\pi}\lim_{\rho\to\beta}\left[\sqrt{R(\beta-\rho)}\ \sigma_{zz}(\rho, 0)\right] = -\frac{G}{1-\nu}\left(\frac{\pi R}{2\beta}\right)^{1/2}\sum_{m=0}^{\infty} Q_{2m}\beta^{2m}$$

$$K^* = \frac{K\lambda\sqrt{\pi}}{GmW_0R^{3/2}} \tag{2.14}$$

where

$$\sigma_{zz}(\rho, 0) = \sigma_{zz}^{(1)}(\rho, 0) + \sigma_{zz}^{(2)}(\rho, 0)$$

The dependence of the quantity K^* on β is presented in Fig. 2 for different values of α($\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\alpha_3 = 0.8$). These results show that for definite relationships between the quantities α and β the origination of compressive ($K < 0$) normal stresses on the continuation of the annular slot is possible.

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